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PERTURBED MOTION OF VISCOELASTIC COLUMNS: A VARIATIONAL APPROACH

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Abstract—In the past, the stability of viscoelastic columns has been analysed by solving the integro-differential equations of equilibrium under static or dynamic type disturbances. The solution of these is usually difficult and the authors intend to provide an alternative formulation of variational type. By using a convolution bilinear form, the operator governing the problem becomes symmetric and a functional, which is stationary at the solution of classical equations, is obtained. This formulation makes it possible both to use the classical approximation methods of the variation calculus and to pose the problem on more natural functional spaces. Some applications show the potential of the Ritz classical spectral method in the numerical solution and introduce the problem of columns subjected to variable load history.

NOTATION

A_{ijkl}, b_{ij}, u_{ij}	matrix components
A, B, C, G, N, P, S, T	linear operators
A, B, C, H, U, U_0, V, W	linear spaces
A, B, C, H, U, U_0, V, W	linear spaces
$D(\cdot)$	domain of a linear operator
d_α, h_α	functions of time variable t
$\gamma_\alpha, \gamma_\alpha^*, \delta_\alpha, \delta_\alpha^*, I_0, I_0^*, I_T, I_T^*$	trace operators
E_1, E_2, ν_2	parameters for three element model
$\mathscr{F}$	functional
j, m	functions of spatial variable x
L	length of the column
p, g, q, u, v, η	functions of spatial variable x and temporal variable t
ϕ_i, ψ_k	series of orthonormal functions
$R(\cdot)$	range of a linear operator
$t, \bar{t}, T$	time variables
u_m	function of the time variable
u_0	parameter for geometric imperfection description
x	spatial variable
X, Y	reference axes
$\Omega, \bar{\Omega}, \partial\Omega$	spatial domain (open, closed, boundary)
$\circledast$	Boltzmann operator.

INTRODUCTION

The stability of a viscoelastic column subjected to axial loads is a problem of remarkable technical interest which has been the object of many studies in the past. A judgement on the stability of the system evolution under axial loads is usually formulated on the basis of the features of the column motion caused by dynamic type (not homogeneous initial conditions) or static type (geometric imperfections or lateral forces) disturbances. The problem is usually posed in its *Eulerian form* by writing the integro-differential type equations of local equilibrium and the related initial and boundary conditions that define the problem. In the past, the evolution of the system was analysed only with reference to simple constraint and constant load. In particular, Dost and Glockner (1985) approached the problem in the case of dynamic disturbances and Szyszkowski and Glockner (1985) in the case of imperfections, reaching solutions in closed form that are useful for understanding the physical question. These solutions were obtained by means of Laplace transforms. Starting with the Euler approach, more general situations require numerical methods which are very complex and laborious and the Laplace transform method does not permit dealing with columns subjected to axial load varying in time.

In this paper the so-called *inverse problem* of the calculus of variations has been dealt with by seeking a functional which is stationary at the solution of the problem in its classical formulation. In this way a double advantage is obtained. The first provides a more extensive formulation to the problem because the functional requires a lower and more natural regularity for the functions describing data and unknowns with respect to classical formulation. The second advantage consists of the possibility of applying the numerical methods usually used in weak formulations.

However the characterization of this functional is not a trivial matter and cannot be obtained in the classical way because the operator ruling the problem is not symmetric with respect to the classical bilinear form.

Historically the problems ruled by non-symmetric operators resisted variational formulation for a long time and the first results are due to Gurtin (1963, 1964) and to Morse and Feshbach (1953), with reference to particular problems. A clearer outlook on the problem was finally obtained thanks to Tonti (1973) who focused the question on the meaning of the operator symmetry and on the fact that this property depends on the particular bilinear form used. This interpretation led to a number of generalizations as in Reddy (1975). Especially interesting is the one proposed by Magri (1974), in which a method for the variational formulation of every linear problem is presented, and the generalization by Tonti himself (1984) in which a general method for non-linear problems is proposed.

In this paper the authors investigate the possibility of obtaining a variational formulation of the problem by means of the convolution bilinear form, with positive results. This bilinear form was previously used for the classical linear dynamic viscoelastic problem, Leitman (1966), however it must be noted that the effect of an axial force makes the operator of the problem considered more complex and substantially different. This creates some difficulty and the symmetry is preserved only if the axial action on the column in the interval $[0, T]$ is a time function symmetric around $T/2$. In some cases of engineering interest the previous condition is satisfied (constant actions). However this limitation can be bypassed by considering a time interval which is twice as long. Consequently the case of variable axial load can be treated avoiding the difficulties related to the Laplace transform method. As previously mentioned, a variational formulation can be obtained by applying Magri's method. This would lead to a fairly complex functional and this substantially motivated the search for an alternative method.

In defining the functional, the authors tried to make the initial and boundary conditions as general as possible. For this purpose it was necessary to use formal notation, together with classical notation whenever this was useful.

In the past, variational methods were rarely applied to the numerical solution of viscoelastic problems and the authors intend to underline the potentialities of these by concluding with two applications of the classical Ritz spectral method. In the first application, the results obtained numerically are compared with a known closed-form solution, while the second application describes the behaviour of columns subjected to a variable load history, finite in time.

THE COLUMN PROBLEM

The column represented in Fig. 1 is considered. Its length is L and its undisturbed motion involves only an axial action $p(x, t)$ and homogeneous initial conditions. The disturbed motion of this column is analysed assuming that strain and displacements are small. The X–Y plane is a symmetry plane, the shear strain is neglected and the cross-sections remain plane after deformation. In the classical form, the problem is characterized by the field equation

$$[jg \circledast u'']'' - [pu']' + m\ddot{u} = q \quad \text{on } \Omega \times (0, T), \tag{1a}$$

where Ω denotes the spatial interval $(0, L)$, the primes the spatial derivatives with respect

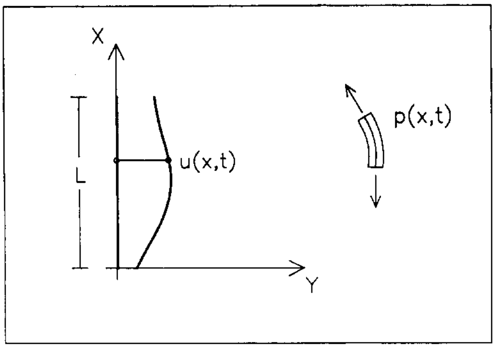

Fig. 1. Geometry of the problem.

to x, the dots those with respect to the time, $\circledast$ the Boltzmann operator†, $u \in U \subseteqq C^{4,2}(\Omega \times (0, T))$‡ the lateral displacements of the axes of the column, $q \in V \subseteqq C^{0,0}(\Omega \times (0, T))$ a disturbance term due to geometric imperfections or to lateral actions, j the moment of inertia of the cross-section, g the relaxation function of the material, p the axial action and m the mass per unit length. It is assumed that the functions p and g are defined on $\bar{\Omega} \times [0, T]$, while j and m are constants in time, i.e. defined on $\bar{\Omega}$; furthermore it is assumed that all these functions have the regularity required to give a sense to eqn (8a). In the following, the authors preferred to obtain the results by means of formal expressions. Equation (8a), in other words the Euler equation of the problem, can be formally written

$$A^*GAu + B^*PBu + C^*MCu = q \quad \text{on } \Omega \times (0, T), \tag{1b}$$

where A, B, C, A^*, B^*, C^* denote the differential operators, P and M the operators that map a function on to the same function multiplied respectively by p and m, and G is the operator involving j, g and the Boltzmann operator defined in footnote †. The differential operators A, A^* and C, C^* differ only because they act on different functional spaces.

The problem is completed by the boundary and initial conditions.

Generally, some of the conditions on the values that u and u' assume on the ends, are assigned (Dirichlet conditions). The symbols γ_α ($\alpha = 1, 2$) denote the operators, that connect u with the value that the same function ($\gamma_1 u = u$) and its derivative ($\gamma_2 u = u'$) assume at the ends. The boundary conditions can be formally posed in the form

$$\gamma_\alpha u = d_\alpha \quad \text{on } \partial\Omega_\alpha \times [0, T], \tag{2}$$

where $\partial\Omega_\alpha$ denotes the set of end-points in which the conditions $\gamma_\alpha u = d_\alpha$ are assigned.

Other Neumann type conditions on the values that the functions $\gamma_1^* u = jg \circledast u''$ (bending moment) and $\gamma_2^* u = [jg \circledast u'']' - pu'$ (shear) assume at the ends are usually assigned. By means of the criterion of the previous case, the following formal equation can be posed:

$$\gamma_\alpha^* u = h_\alpha \quad \text{on } \partial\Omega_\alpha^* \times [0, T]. \tag{3}$$

A total of four boundary conditions, some of the former and some of the latter types,

†The Boltzmann operator is the linear operator characteristic of the viscoelastic constitutive law, as introduced by Leitman and Fisher (1973). This is defined by the following equality:

$$g(t) \circledast u(t) = g(0)u(t) + \dot{g}(t) * u(t) = g(0)u(t) + \int_0^t \dot{g}(t-\tau)u(\tau)\,d\tau,$$

where $g(t) \in AC[0, \infty)$ (AC = absolutely continuous).

‡The form $C^{m,n}(\Omega \times (0, T))$ denotes the space of continuous functions defined on $\Omega \times (0, T)$ for which both the derivatives with respect to $x \in \Omega$ of order less than or equal to m and the derivatives with respect to time of order less than or equal to n, are continuous.

must be assigned. At least two conditions must be of the former type to avoid non-uniqueness of the solution as a result of rigid motions.

Finally the problem is completed with the initial conditions (Cauchy conditions) that assign the value of u and its time derivative $\dot{u}$ at the initial instant $t = 0$:

$$\begin{aligned} I_0 u &= u_0 \\ I_0^* u &= \dot{u}_0 \end{aligned} \quad \text{on } \bar{\Omega} \times t = 0. \tag{4}$$

VARIATIONAL FORMULATION

The linear problem considered has the form

$$Na = b, \tag{5}$$

where $N{:}D(N) \to R(N)$, $D(N)$ is a subset of a linear space A, and $R(N)$ is dense in a second linear space B. The operator N consists of an integro-differential operator, mapping functions "a" into functions "b" defined on an open set Ω (formal part), and of boundary conditions mapping functions "a" into functions defined on the boundary $\partial\Omega$ (trace operators). The authors intend to deal with the so-called *inverse problem* of the calculus of variations, by seeking a functional $\mathscr{F}(a) : D(N) \to \mathbb{R}$ which is stationary at the solution of problem (5), i.e.

$$\delta\mathscr{F}(a;\eta) = 0 \quad \forall \eta : (a+\eta) \in D(N) \Leftrightarrow Na = b. \tag{6}$$

This is a classical topic of applied mathematics and a historical review can be found in Tonti (1984). In the linear case the following result can be shown: if a form $\langle a, b\rangle : A \times B \to \mathbb{R}$ is bilinear (i), non-degenerate (ii) (i.e. $\langle a, b\rangle = 0 \ \forall a \in A \Rightarrow b = 0$ and $\langle a, b\rangle = 0 \ \forall b \in B \Rightarrow a = 0$) and continuous (iii) and if the linear operator N is symmetric, in the sense that it satisfies the condition

$$\langle a_2, Na_1\rangle = \langle a_1, Na_2\rangle \quad \forall a_1, a_2 \in A, \tag{7}$$

then the functional sought exists and possesses the following form:

$$\mathscr{F}(a) = 1/2\langle a, Na\rangle - \langle a, b\rangle. \tag{8}$$

It is useful to note that the usual canonical bilinear form derived from the scalar product is only one of the infinite forms that can be adopted and does not make the analysed operator symmetric.

The formal part of the linear operators described in the rest of this paper are written in the form $N = T^*ST$. The symmetry of these can be proved by showing that T^* is the *adjoint* of T, i.e. $\langle T^*a, b\rangle = \langle a, Tb\rangle$ and by showing that S is symmetric.

In the problem considered, the presence of terms related to the axial force in the differential equation (B^*PB) and in some boundary conditions ($-pu'$ in $\gamma_2^* u$), makes the linear operator substantially different from the operator ruling the dynamic viscoelastic problem, which involves only A^*GA and C^*MC (with correspondent boundary or initial conditions) and admits a variational formulation by means of the convolution bilinear form (Leitman, 1966; Tonti, 1973). The aim is to show if, and under which assumptions, the convolution bilinear form makes it possible to obtain the required symmetry even in the presence of the operatorB^*PB and of the particular boundary conditions $\gamma_2^* u = h_2$.

The following bilinear forms are introduced:

—bilinear form between functions defined on $\Omega \times (0, T)$

$$\langle u, v\rangle_{\mathrm{I}} = \int_{\Omega} u * v \,\mathrm{d}\Omega = \int_0^L \int_0^T u(x, T-t)v(x, t)\,\mathrm{d}t\,\mathrm{d}x, \tag{9}$$

—bilinear form between functions defined on $\partial\Omega_\alpha \times [0, T]$

$$\langle u, v\rangle_{\mathrm{II}_\alpha} = \pm u * v = \pm \int_0^T u(T-t)v(t)\,\mathrm{d}t, \tag{10}$$

where the positive sign holds at $x = L$ and the negative at $x = 0$; in a similar manner, the bilinear form $\langle u, v\rangle_{\mathrm{II}_\alpha^*}$ between functions defined on $\partial\Omega_\alpha^* \times [0, T]$, is defined;

—bilinear form between functions defined on $\bar{\Omega}$

$$\langle u, v\rangle_{\mathrm{III}} = \int_{\Omega} uv \,\mathrm{d}\Omega = \int_0^L u(x)v(x)\,\mathrm{d}x. \tag{11}$$

It is well known that the previous bilinear forms are non-degenerate between functional spaces for the Titchmarsh's theorems (Yosida, 1980).

For the variational formulation, A^*, B^*, C^* must be formal adjoints of A, B, C, and G, P, M must be symmetric. The relation of formal adjoint between A^*, C^* and A, C with respect to the bilinear form $\langle u,v\rangle_{\mathrm{I}}$ is known from the variational formulation of the dynamic viscoelastic problem; it can be shown that even $B^* = -\partial/\partial x$ is the formal adjoint of $B = \partial/\partial x$. In fact, by integrating by parts

$$\langle Bu_1, u_2\rangle_{\mathrm{I}} = \int_{\Omega} u_1' * u_2 \,\mathrm{d}\Omega = [u_1 * u_2]_0^L - \int_{\Omega} u_1 * u_2' \,\mathrm{d}\Omega = \int_{\Omega} u_1^*(-u_2')\,\mathrm{d}\Omega = \langle u_1, B^* u_2\rangle_{\mathrm{I}}, \tag{12}$$

where the terms on the boundary disappear if the trace of u is null, as per the definition of formal adjoint (Oden, 1979).

With respect to this bilinear form, the operators G and M are also symmetric. The symmetry of M is trivial while the symmetry of G can be shown by means of a change in the integral variables and by introducing the Heaviside function (Tonti, 1973).

For the variational formulation the operator P must also be symmetric. In general, the convolution form does not permit obtaining this result because the following equations hold:

$$\langle Pu_1, u_2\rangle = \int_0^L \int_0^T p(x, T-t)u_1(x, T-t)u_2(x, t)\,\mathrm{d}x\,\mathrm{d}t, \tag{13a}$$

$$\langle u_1, Pu_2\rangle = \int_0^L \int_0^T p(x, t)u_1(x, T-t)u_2(x, t)\,\mathrm{d}x\,\mathrm{d}t. \tag{13b}$$

An equivalence between the two terms is true for every pair u_1, u_2 only if $p(x, t) = p(x, T-t)$ in $[0, T]$. Taking into account that $p(x, t)$ represents the axial action of the undisturbed motion, it can be concluded that the required symmetry holds in many practical problems in which $p(x, t)$ is constant in time. However, in those cases in which $p(x, t)$ varies in the temporal interval $[0, T]$, the limitation can be bypassed by studying the phenomenon in the interval $[0, 2T]$ under an axial action $\bar{p}(x, t)$ such that

$$\bar{p}(x, t) = p(x, t) \qquad \text{on } \bar{\Omega} \times [0, T], \tag{14a}$$

$$\bar{p}(x, t) = p(x, 2T-t) \quad \text{on } \bar{\Omega} \times (T, 2T]. \tag{14b}$$

In order to take into account the boundary and initial conditions, it is necessary to introduce some duality relations between the spaces in which d_α, h_α, u_0, $\dot{u}_0$ are defined and the space of functions u. This can be done by defining some suitable trace operators that project u on the boundary and allow the required symmetries to be found.

With regards to the boundary conditions, the formula of integration by parts

$$\int_\Omega ((jg \circledast u_1'')'' - (pu_1')') * u_2 \, d\Omega = \int_\Omega jg \circledast u_1'' * u_2'' + pu_1' * u_2' \, d\Omega + [((jg \circledast u_1'')' - pu_1') * u_2]_0^L - [jg \circledast u_1'' * u_2']_0^L \quad (15)$$

suggests putting the time functions $\gamma_\alpha u$ on the points $\partial\Omega_\alpha$ in duality with the functions obtained from $u \in U$ by means of the operators $\delta_\alpha^* u$ defined as

$$\delta_1^* u = (jg \circledast u'')' - pu' \quad \text{on } \partial\Omega_1 \times [0, T], \quad (16a)$$

$$\delta_2^* u = -jg \circledast u'' \quad \text{on } \partial\Omega_2 \times [0, T], \quad (16b)$$

and putting $\gamma_\alpha^* u$ in duality with the functions obtained by means of the following operators $\delta_\alpha u$:

$$\delta_1 u = u' \quad \text{on } \partial\Omega_1^* \times [0, T], \quad (17a)$$

$$\delta_2 u = -u \quad \text{on } \partial\Omega_2^* \times [0, T]. \quad (17b)$$

Once this has been established, the equivalence of eqn (15), which is useful for demonstrating the variational theorem, can be re-written in the following formal way:

$$\langle A^* GAu_1, u_2 \rangle_{\mathrm{I}} + \langle B^* PBu_1, u_2 \rangle_{\mathrm{I}} + \Sigma_\alpha \langle \gamma_\alpha^* u_1, \delta_\alpha u_2 \rangle_{\mathrm{II}_\alpha^*} = \langle GAu_1, Au_2 \rangle_{\mathrm{I}} + \langle PBu_1, Bu_2 \rangle_{\mathrm{I}} + \Sigma_\alpha \langle \delta_\alpha^* u_1, \gamma_\alpha u_2 \rangle_{\mathrm{II}_\alpha}. \quad (18)$$

With regards to the initial conditions, the formula of integrations by parts

$$\int_\Omega m\ddot{u}_1 * u_2 \, d\Omega = \int_\Omega m\dot{u}_1 * \dot{u}_2 \, d\Omega + \int_\Omega [m\dot{u}_1(t) u_2(T-t)]_0^T \, d\Omega \quad (19)$$

suggests introducing the operators

$$\left.\begin{aligned} I_T^* u &= m\dot{u}(x, T) \quad (20a) \\ I_T u &= mu(x, T) \quad (20b) \end{aligned}\right\} \text{on } \bar{\Omega} \times t = T.$$

Equation (19) can be rewritten in the following formal way:

$$\langle C^* MCu_1, u_2 \rangle_{\mathrm{I}} + \langle I_0^* u_1, I_T u_2 \rangle_{\mathrm{III}} = \langle MCu_1, Cu_2 \rangle_{\mathrm{I}} + \langle I_T^* u_1, I_0 u_2 \rangle_{\mathrm{III}}. \quad (21)$$

On the basis of the results obtained, it is possible to demonstrate the variational theorem in formal terms.

Theorem. The differential problem consisting of eqns (1)–(4) *is assigned. If* $p(x, t) = p(x, T-t) \; \forall \, t \in [0, T]$ *then the functions u satisfying the differential problem coincide with the critical point of the functional*

$$\mathscr{F}(u) = 1/2[\langle GAu, Au \rangle_{\mathrm{I}} + \langle PBu, Bu \rangle_{\mathrm{I}} + \langle MCu, Cu \rangle_{\mathrm{I}} - \langle q, u \rangle_{\mathrm{I}}] + \Sigma_\alpha \langle \gamma_\alpha u - d_\alpha, \delta_\alpha^* u \rangle_{\mathrm{II}_\alpha} - \Sigma_\alpha \langle h_\alpha, \delta_\alpha u \rangle_{\mathrm{II}_\alpha^*} + \langle I_0 u - u_0, I_T^* u \rangle_{\mathrm{III}} - \langle \dot{u}_0, I_T u \rangle_{\mathrm{III}}. \quad (22)$$

Proof. (a) It will be shown that if u satisfies eqns (1)–(4) then the functional $\mathscr{F}(u)$ is stationary at u. The differential of the functional with respect to a variation η assumes the following form:

$$
\begin{aligned}
\delta\mathscr{F}(u;\eta) = 1/2[&\langle GA\eta, Au\rangle_{\mathrm{I}} + \langle GAu, A\eta\rangle_{\mathrm{I}} + \langle PB\eta, Bu\rangle_{\mathrm{I}} \\
&+ \langle PBu, B\eta\rangle_{\mathrm{I}} + \langle MC\eta, Cu\rangle_{\mathrm{I}} + \langle MCu, C\eta\rangle_{\mathrm{I}} - \langle q, \eta\rangle_{\mathrm{I}}] \\
&+ \Sigma_\alpha[\langle \gamma_\alpha u - d_\alpha, \delta_\alpha^* \eta\rangle_{\mathrm{II}_\alpha} + \langle \gamma_\alpha \eta, \delta_\alpha^* u\rangle_{\mathrm{II}_\alpha}] - \Sigma_\alpha \langle h_\alpha, \delta_\alpha \eta\rangle_{\mathrm{II}_\alpha^*} \\
&+ \langle I_0 u - u_0, I_T^* \eta\rangle_{\mathrm{III}} + \langle I_0 \eta, I_T^* u\rangle_{\mathrm{III}} - \langle \dot{u}_0, I_T \eta\rangle_{\mathrm{III}}.
\end{aligned}
\tag{23}
$$

For the symmetry of G, P, M and using the relations of eqns (18) and (21), the following can be obtained

$$
\begin{aligned}
\delta\mathscr{F}(u;\eta) = &\langle A^*GAu + B^*PBu + C^*MCu - q, \eta\rangle_{\mathrm{I}} + \Sigma_\alpha \langle \gamma_\alpha u - d_\alpha, \delta_\alpha^* \eta\rangle_{\mathrm{II}_\alpha} \\
&+ \Sigma_\alpha \langle \gamma_\alpha^* u - h_\alpha, \delta_\alpha \eta\rangle_{\mathrm{II}_\alpha^*} + \langle I_0 u - u_0, I_T^* \eta\rangle_{\mathrm{III}} + \langle I_0^* u - \dot{u}_0, I_T \eta\rangle_{\mathrm{III}},
\end{aligned}
\tag{24}
$$

such that $\delta\mathscr{F}(u;\eta) = 0$ for every variation η when u is a solution of eqns (1), (2), (3), (4), and therefore a critical point exists at u.

(b) And, vice versa, if the functional is stationary at u then u satisfies eqns (1)–(4). According to the form that the functional differential assumes in eqn (24) and because the bilinear form is non-degenerate [property (ii)], it can be directly concluded that if the differential of eqn (24) is null for every variation η then the former term of every bilinear form must be null, i.e. eqns (1)–(4) are satisfied. □

It can generally be noted that the linear operator is not positive with respect to the adopted bilinear form so that the stationary point does not necessarily represent an extreme (minimum or maximum) of the functional.

Essential simplifications are often possible in the expression of $\mathscr{F}(u)$ [eqn (22)]. Usually, when the fundamental period of free vibration of the structure is much smaller than the relaxation time of the material, it is reasonable to neglect the inertia forces, (Hoff, 1958). In this case the functional becomes simpler because the terms involving C^*MC and the initial conditions disappear. Furthermore if u is forced to belong to a subset $U_0 \subseteqq U$ where U_0 denotes the space of functions that satisfy the Dirichlet type boundary conditions, the term related to $\gamma_\alpha u - d_\alpha$ is also null. In this last case, U_0 is no longer a linear space unless the Dirichlet conditions are homogeneous.

In conclusion, it can be noted that, as usually occurs in variational formulations, the functional $\mathscr{F}(u)$ can be used to define the problem in a more general way, closer to the physical question. In fact, $\mathscr{F}(u)$ can be defined for $u \in U_0 \subseteqq H^{2,1}(\Omega \times (0, T))$†; a lower and more natural degree of regularity is consequently required for u that must have only one continuous spatial derivative, which is equivalent to requiring the continuity of the displacement field in the column. The data space is also enlarged and the form is defined for $q \in V \subseteqq H^{-2,-1}(\Omega \times (0, T))$, and consequently also for concentrated actions and discontinuous or impulsive load history. Similar arguments make it possible to conclude that a lower regularity with respect to the classical requirements suffices for j, p and m too.

APPLICATION I

In the case of constant actions, the system progresses in time fitting a very regular curve, which makes the numerical solution using the Ritz classical spectral method, a very profitable tool. This can be shown by solving an analytically known problem, numerically. In particular, a simple case solved in closed form by Szyszkowski and Glockner (1985) is considered. This refers to a column pinned at the ends, where j is a constant, subjected to

† The form $H^{m,n}(\Omega \times (0, T))$ denotes the space of functions defined on $\Omega \times (0, T)$ for which both the derivatives with respect to x of order less than or equal to m and the derivatives with respect to time of order less than or equal to n are on $L^2(\Omega \times (0, T))$.

an axial action p constant in time and in space (Figure 2). A static analysis is carried out and the disturbance consists of a geometric imperfection of the axis of the beam described by the sinusoidal function $u_0 \sin(\pi x/L)$, comparable to a lateral distributed action $q(x,t) = pu_0\pi^2/L^2 \sin(\pi x/L)$ (the sign is not important). The behaviour of the material is modelled by a three-element model, so that the kernel $g(t)$ is described by the following expressions:

$$g(0) = E_1; \tag{25a}$$

$$\dot{g}(t) = -\frac{E_1^2}{\nu_2}\exp\left(-\frac{E_1+E_2}{\nu_2}t\right), \tag{25b}$$

where $E_1 = E_2 = 30\,000$ MPa and $\nu_2 = 1.0 \times 10^6$ MPa days.

The numerical solution can be obtained by assuming that an orthonormal series $\phi_i(x)\psi_k(t)$ complete in U_0 (functions satisfying the Dirichlet boundary conditions) exists. The approximate problem can be solved in the finite-dimensional subspace W produced by the terms of the series such that $i \leqslant n$ and $k \leqslant m$. The projection of $u(x,t)$ in such a subspace has the form $\tilde{u}(x,t) = u_{ik}\phi_i(x)\psi_k(t)$ where the repeated indices denote summation, and the coefficients u_{ik} are the usually scalar product $(u, \phi_i\psi_k)$. In this subspace W, the functional (22) depends on the terms u_{ik} and assumes the following form:

$$\mathscr{F}(\tilde{u}) = \{1/2[\langle GA\phi_i\psi_k, A\phi_r\psi_s\rangle_1 + \langle PB\phi_i\psi_k, B\phi_r\psi_s\rangle_1]u_{ik} - \langle q, \phi_r\psi_s\rangle_1\}u_{rs}. \tag{26}$$

The stationary condition requires the simultaneous annulment of the partial derivatives and, by means of the positions

$$A_{ikrs} = [\langle GA\phi_i\psi_k, A\phi_r\psi_s\rangle_1\langle PB\phi_i\psi_k, B\phi_r\psi_s\rangle_1], \tag{27a}$$

$$b_{rs} = \langle q, \phi_r\psi_s\rangle_1 \tag{27b}$$

can be written as

$$A_{ikrs}u_{ik} - b_{rs} = 0 \tag{28}$$

for every $r \leqslant n, s \leqslant m$.

The solution can be provided assuming the following orthonormal sinusoidal series for the variable x:

$$\phi_i(x) = \sqrt{\frac{2}{L}}\sin\frac{i\pi x}{L} \quad i = 1, \ldots, n \tag{29}$$

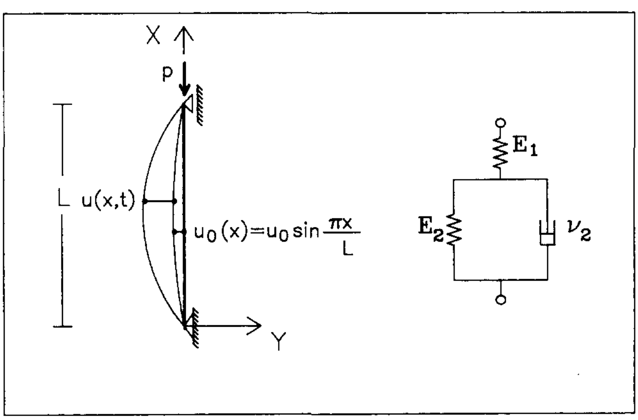

Fig. 2. Imperfect column with three-element model.

Table 1. Ratio $u_m(t)/u_0$ for (a) $p/p_E = 0.4$ and (b) $p/p_E = 0.6$

(a)

t	2 Terms	3 Terms	4 Terms	5 Terms	Exact
0	1.856	1.685	1.667	1.666	1.666
20	2.271	2.264	2.270	2.270	2.270
40	2.685	2.761	2.765	2.765	2.765
60	3.100	3.175	3.170	3.170	3.170
80	3.514	3.507	3.501	3.502	3.502
100	3.929	3.757	3.774	3.773	3.773

(b)

t	2 Terms	3 Terms	4 Terms	5 Terms	Exact
0	−0.183	2.915	2.454	2.505	2.500
20	4.853	4.977	5.143	5.122	5.123
40	9.890	8.526	8.655	8.667	8.665
60	14.926	13.563	13.434	13.446	13.447
80	19.963	20.087	19.921	19.900	19.900
100	25.000	28.098	28.559	28.609	28.612

and the normalized Legendre series for the variable t:

$$\psi_0(t) = \frac{1}{\sqrt{T}} \qquad k = 0,$$

$$\psi_k(t) = \sqrt{\frac{2k-1}{T}}\,\frac{1}{k!2^k}\,\frac{d^k}{dx^k}((2t/T)^2-1) \quad k = 1,\ldots,n. \tag{30}$$

It must be noted that the particular form of q and the orthogonality of the $\phi_i(x)$ and of their derivatives makes all the coefficients b_{rs} with $r \neq 1$ and A_{ikrs} with $i \neq r$ null.

It can be shown that the terms of the normalized Legendre series preserve their orthogonality also with respect to the convolution bilinear form, therefore the terms $\langle\psi_k, \psi_s\rangle_1 = \psi_k * \psi_s$ are null if $k \neq s$, even if the composition of functions with the same index no longer provide their norm. In particular the bilinear form provides $\langle\psi_k, \psi_s\rangle_1 = \psi_k * \psi_s = \delta_{ks}(-1)^k$ where δ_{ks} = Kronecker delta (in this expression, the repeated index k does not denote a summation).

In Table 1, the amplification of the disturbance is reported by means of the ratio between the maximum displacement $u_m(t)$ and the amplitude of the disturbance u_0 obtained for two meaningful values of the load. The ratio between axial load p and the Euler critical load p_E equal to 0.4 is examined in the former case (column with asymptotic bounded displacement) and the ratio 0.6 in the latter case (column with asymptotic unbounded displacement). The solutions obtained with the first terms of the Legendre series are reported.

APPLICATION 2

The second application shows the effect of axial load histories variable and finite in time. This problem is not developed in the literature and cannot be approached in a useful manner by using the Laplace transform method. The example intends to underline only some features of this complex problem.

The column of the previous application is considered and the axial load history is described by the following function:

$$p(x,t) = p_m \sin\frac{\pi t}{n\bar{t}} \quad \text{on } [0, n\bar{t}], \tag{31a}$$

$$p(x,t) = 0 \qquad \text{on } (n\bar{t}, \infty), \tag{31b}$$

where $\bar{t} = v_2/(E_1+E_2)$ denotes the characteristic relaxation time $(g(t+\bar{t})/g(t) = e^{-1})$, p_m is the maximum axial load and n is an integer.

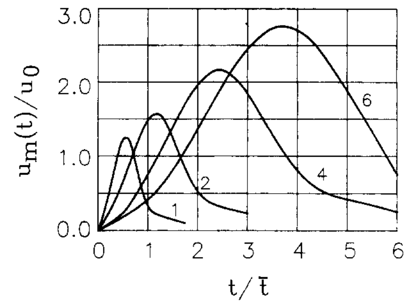

Fig. 3. Ratio u_m/u_0 for different load durations.

Figure 3 shows the ratio $u_m(t)/u_0$ for loads with different durations ($n = 1, 2, 4, 6$) assuming $p_m/p_E = 0.5$. This value of axial action represents the critical viscous load in the case of constant load (Szyszkowski and Glockner, 1985) and identifies the boundary between bounded and unbounded asymptotic behaviour. It can be noted that the load duration notably affects the maximum value of displacement and this may be acceptable if the period of loading is sufficiently brief.

Figure 4 shows the ratio $u_m(t)/u_0$ for $n = 4$ and for different values of the ratio p_m/p_E (0.3, 0.4, 0.5, 0.6). The dotted curve describes the deformation of an elastic column with an elastic modulus equal to the initial modulus E_1 and $p_m/p_E = 0.5$. It is interesting to observe that the maximum displacement value occurs after the application of the maximum load p_m and this delay is scarcely affected by the axial load. As expected, the correlation between displacements and axial load is non-linear.

This simple example shows that this problem presents interesting features and the authors believe that a better understanding requires a major investigation into the work developed in this demonstrative application.

CONCLUSIONS

The purpose of this study was to provide a variational formulation of the problem of viscoelastic columns subjected to axial loads, as an alternative to the classical Euler form, whose solution is often difficult.

This was achieved by adopting a convolution bilinear form instead of the canonic bilinear form with respect to which the operator of the problem is not symmetric. The convolution form leads to a functional that has a stationary point (not necessarily extreme) at the solution of the problem in the classical form. This functional was obtained under very general boundary and initial conditions.

Consequently it has been possible to extend the usual advantages that make the variational formulation preferable over the classical differential formulation, for this type of problem; in fact it is possible to operate on larger and more physically motivated

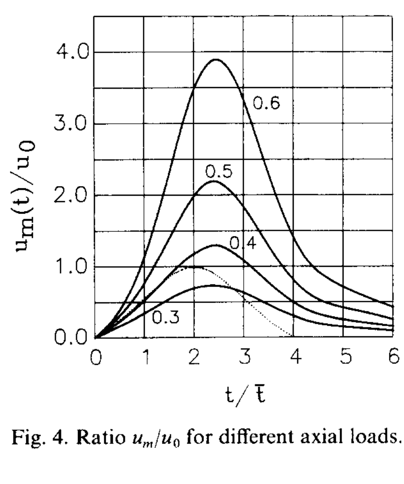

Fig. 4. Ratio u_m/u_0 for different axial loads.

functional spaces and to use the classical methods of approximation of the calculus of variations, rarely applied to viscoelastic problems.

In a first numerical application the effectiveness of the classical Ritz spectral method was demonstrated in the case of constant axial actions by numerically solving a problem whose analytical solution is known. The effectiveness of this method is substantially due to the regularity of the function describing the progress in time of the system. A second application shows the behaviour of viscoelastic columns subjected to a finite load history; this problem possesses interesting features and cannot be approached by the classic Laplace transform method.

REFERENCES

Dost, S. and Glockner, P. G. (1985). The stability of viscoelastic perfect columns: a dynamic approach. *Int. J. Solids Structures* **21**(6), 545–559.

Gurtin, M. E. (1963). Variational principles in the linear theory of viscoelasticity. *Arch. Rat. Mech. Anal.* **13**(3), 179–191.

Gurtin, M. E. (1964). Variational principles for linear initial value problems. *Q. Appl. Math.* **22**, 252–256.

Hoff, N. J. (1958). A survey of the theories of creep buckling. *Proceedings of the Third U.S. National Congress of Applied Mechanics.* ASME, New York.

Knops, R. J. and Wilkes, E. W. (1973). Theory of elastic stability. In *Handbuch der Physik*, Vol. VIa/3, p. 125. Springer, Berlin.

Leitman, J. M. and Fisher, G. M. C. (1973). The linear theory of viscoelasticity. In *Handbuch der Physik*, Vol. VIa/3, pp. 10–31. Springer, Berlin.

Leitman, M. J. (1966). Variational principles in the linear dynamic theory of viscoelasticity. *Q. Appl. Math.* **24**(1), 37–46.

Magri, F. (1974). Variational formulations for every linear equation. *Int. J. Engng Sci.* **12**, 537–549.

Morse, P. M. and Feshbach, H. (1953). *Methods of Theoretical Physics*, Vol. 1. McGraw-Hill, Maidenhead.

Oden, J. T. (1979). *Applied Functional Analysis.* Prentice-Hall, Englewood Cliffs, NJ.

Reddy, J. N. (1975). A note on mixed variational principles for initial value problems. *Q. J. Mech. Appl. Math.* **28**, 123–132.

Szyszkowski, S. and Glockner, P. G. (1985). The imperfect linearly viscoelastic column. *Int. J. Engng Sci.* **23**(10), 113–1120.

Tonti, E. (1973). On the variational formulation for linear initial value problems. *Annali di matematica pura ed applicata* (IV), **XCV**, 331–360.

Tonti, E. (1984). Variational formulation for every nonlinear problem. *Int. J. Engng Sci.* **22**(11/12), 1343–1371.

Yosida, K. (1980). *Functional Analysis.* Springer, Berlin.